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Abstract Differential Equations with Almost-Periodic Solutions*

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In this paper we present some quite simple results concerning almost-periodic solutions of abstract differential equations. We start with a general proposition about linear equations, then we establish some new facts about bounded or relatively compact solutions which become almost-periodic; finally, we study "separation from 0 properties" of non-trivial almost-periodic solutions for equations with bounded or unbounded operator coefficients. © 1985 Academic Press, Inc.

1

We begin with a quite natural assertion related to Theorem 5.12 in [5, p. 93]. Let X be a Banach space over a scalar field, and A; $D(A) \subset X \to X$ be a linear closed operator of domain $D(A) \subset X$ and range in X too. We shall use the notation $AP(X)$ for the (Banach) space of (continuous) almost-periodic functions, $\mathbb{R} \to X$ equipped with the uniform norm

$$\| f(\cdot)\|_{AP(X)} = \sup_{t \in \mathbb{R}} \| f(t)\|_X.$$

For any $f \in AP(X)$ denote by $\exp(f) = (\lambda_n)_1^\infty$ the countable set of those real numbers λ such that

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T e^{-i\lambda t} f(t)\, dt \text{ is} \neq \theta.$$

Given now any subset Λ of the real line, let us call $AP_\Lambda(X)$ the set of those $f \in AP(X)$, such that $\exp(f) \subseteq \Lambda$; it is obviously a closed linear subspace of $AP(X)$. Let us state now the following

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THEOREM 1. *Let us assume that for any $f \in AP_A(X)$ there exists one and only one solution $x_f \in AP_A(X)$, such that $dx_f/dt \in AP_A(X)$, of the (abstract) differential equation $dx_f/dt = Ax_f + f$. Then, the mapping $f \to x_f$ is linear and continuous, $AP(X) \to AP(X)$.*

Proof. The linearity follows obviously from uniqueness, and from the linearity of the space $AP_A(X)$.

In order to establish the continuity of the mapping we shall first introduce a new space $AP^1_A(X) = \{f, \mathbb{R} \to X$ such that f and df/dt belong to $AP_A(X)\}$. This is again a closed linear subspace of $AP_A(X)$ as easily seen, under the norm

$$\| f \|_{AP^1_A(X)} = \sup_{t \in \mathbb{R}} \| f(t) \|_X + \sup_{t \in \mathbb{R}} \| f'(t) \|_X.$$

Next we consider the space $AP^{1,A}_A(X)$ consisting of the functions $f \in AP^1_A(X)$, with the supplementary property that $f(t) \in D(A)$ $\forall t \in \mathbb{R}$ and $Af(t) \in AP_A(X)$, $t \in \mathbb{R}$. This is again a linear closed subspace of $AP^1_A(X)$ under the norm

$$\| f \|_{AP^{1,A}_A(X)} = \| f \|_{AP^1_A(X)} + \sup_{t \in \mathbb{R}} \| (Af)(t) \|_X$$

as is readily seen.

We are now able to define a mapping T from (whole) $AP^{1,A}_A(X)$ into $AP_A(X)$ by means of the relation $Tu = du/dt - Au$, $\forall u \in AP^{1,A}_A(X)$. By the assumptions of the theorem T is a surjective (onto) mapping, that is, for any $f \in AP_A(X)$ there is $x(t) \in AP^{1,A}_A(X)$, such that $dx/dt - Ax = f$. It is also an injective (one–one) mapping due to the uniqueness hypothesis. T is also continuous, because of the estimate

$$\| Tx \|_{AP_A(X)} = \sup_{t \in \mathbb{R}} \left\| \frac{dx(t)}{dt} - Ax(t) \right\|_X \leqslant \sup_{t \in \mathbb{R}} \| x'(t) \|_X + \sup_{t \in \mathbb{R}} \| Ax(t) \|_X$$
$$\leqslant \| x \|_{AP^{1,A}_A(X)}.$$

By a theorem of Banach, the inverse mapping T^{-1} (acting from $AP_A(X)$ into $AP^{1,A}_A(X)$, by formula $T^{-1}f = x_f$) is also continuous, so that

$$\| x_f \|_{AP^{1,A}_A(X)} \leqslant C \, \| f \|_{AP_A(X)},$$

which is stronger than what we wished to establish.

2

In this section we establish almost-periodicity of all bounded solutions for a differential equation with null right-hand side and of all relatively compact solutions for differential equations with almost-periodic right-hand side (a previous particular case appeared in [9]). Again let X be a (general) Banach space and $\mathscr{A}$, $\mathscr{D}(\mathscr{A}) \subset X \to X$ be a linear closed operator, densely defined on the linear subspace $\mathscr{D}(\mathscr{A})$, and which is the infinitesimal generator of a strongly almost-periodic group of bounded linear operators $G(t)$, $t \in \mathbb{R} \to \mathscr{L}(X)$ (as in [10]). Next, let X_0 be a finite-dimensional subspace of X and A be a linear continuous mapping of X into X_0 (that is, an operator of finite rank). We shall prove the following

THEOREM 2. *Let us assume that $\mathscr{A}$ commutes with the projection P of X onto X_0 and that $\mathscr{D}(\mathscr{A}) \cap X_0$ is dense in X_0. Then, if $x(t)$, $R \to \mathscr{D}(\mathscr{A})$ is a solution of the equation $dx/dt = (\mathscr{A} + A)\, x(t)$, $t \in \mathbb{R}$, which is bounded over $\mathbb{R}$, it is almost-periodic,* $\mathbb{R} \to X$. (See [6, p. 49] for definition of P; remark that if $x \in \mathscr{D}(\mathscr{A}) \cap X_0$, then $\mathscr{A}x \in X_0$: in fact, $x \in X_0 \Rightarrow Px = x$ and $\mathscr{A}x = \mathscr{A}Px = P\mathscr{A}x \in X_0$.)

Proof. Let $Q = I - P$ (thus $Q^2 = Q$). Remark that

$$\frac{d}{dt}(Qx)(t) = Q\frac{dx}{dt} = Q\mathscr{A}x(t) + QAx(t).$$

Actually QA is the null-operator, as easily seen, and Q commutes with operator $\mathscr{A}$. It follows that $(d/dt)(Qx)(t) = \mathscr{A}(Qx)(t)$, $t \in \mathbb{R}$, and the general solution of this equation is expressed by the relation $(Qx)(t) = G(t)(Qx)(0)$, $t \in \mathbb{R}$ (see, for instance, Lemme 1 in [11, p. 95]). Thus, $(Qx)(t)$ is in $AP(X)$. Next consider the function $(Px)(t)$, whose range is in X_0. We see that

$$\begin{aligned}\frac{d}{dt}(Px)(t) = P\frac{dx}{dt} &= \mathscr{A}(Px)(t) + PA((Px)(t) + (Qx)(t)) \\ &= (\mathscr{A} + PA)(Px)(t) + PA(G(t)(Qx)(0)),\end{aligned}$$

or, denoting $(Px)(t) = y(t)$, and remembering that $A \in \mathscr{L}(X, X_0)$, it is $(d/dt)\, y(t) = (\mathscr{A} + PA)\, y(t) + f(t)$ where $f(t) = A(G(t)(Qx)(0))$ is a.p. from $\mathbb{R}$ to X_0, and $y(t)$ is bounded, $\mathbb{R} \to X_0$. Here $\mathscr{A} + PA = \mathscr{A} + A$ restricted to $X_0 \cap \mathscr{D}(\mathscr{A})$; this is a linear subspace of X_0, hence a closed subspace [6, Corollary I-4-10], and it coincides with X_0 by one of the assumptions. Thus $\mathscr{A}$ is a linear operator $X_0 \to X_0$, hence belongs to $\mathscr{L}(X_0)$. By a well-known theorem (see, for instance, [3, Theorem 4.2]), $y(t) \in AP(X_0)$, hence $x(t) = (Qx)(t) + y(t) \in AP(X)$. Q.E.D.

We end this section by considering the equation with a.p. right-hand side. We prove the following

THEOREM 3. *Let $g(t) \in AP(X)$, while $\mathscr{A}$ and A verify the same assumptions as in Theorem 2. Let $x(t)$, $\mathbb{R} \to \mathscr{D}(\mathscr{A})$ be a solution of the differential equation*

$$\frac{dx}{dt}(t) = (\mathscr{A} + A)\, x(t) + g(t);$$

then, if $x(t)$ has relatively compact range in X it is almost-periodic, and this is also true when X is a Hilbert space and $x(t)$ has a bounded range in it.

Proof. Introducing the projections P and Q as above we see first that

$$\frac{d}{dt}(Qx)(t) = \mathscr{A}(Qx)(t) + (Qg)(t).$$

The range of $(Qx)(t)$ is also relatively compact in X, while Qg is in $AP(X)$. From Theorem 3.2 in [11] we deduce $Qx \in AP(X)$ (when X is a Hilbert space, boundedness of x suffices, as easily seen).

Next, one considers the function Px, $\mathbb{R} \to X_0$ and one readily gets the relation

$$\frac{d}{dt}(Px)(t) = (\mathscr{A} + A)(Px)(t) + AQx(t) + (Pg)(t).$$

This is a differential equation in X_0, where $\mathscr{A} + A \in \mathscr{L}(X_0)$, while the sum $AQx + Pg$ belongs to $AP(X_0)$. Again, from boundedness of Px, $\mathbb{R} \to X_0$ we find $Px \in AP(X)$.

3

In this section we study equations of the form $dx/dt = Ax + f$ where A is an operator of simplest type or a nilpotent operator. Remember (see [4]) that an operator $A \in \mathscr{L}(X)$ (X is again a Banach space) is said to be of simplest type if it admits a representation of the form $A = \sum_{j=1}^{n} \lambda_j P_j$, where the λ_j's are mutually distinct complex numbers and the operators P_j ($j = 1, 2, \ldots, n$) form a complete system ($\sum_{j=1}^{n} P_j = I$) of pairwise disjoint projections in X (i.e., $P_j P_k = \delta_j^k P_k$, $j, k = 1, 2, \ldots, n$). We can state

PROPOSITION 1. *If $x(t)$, $\mathbb{R} \to X$ is a bounded solution of the (homogeneous) equation $dx/dt = Ax$, then $x(t) \in AP(X)$.*

Proof. It is in fact

$$P_k \frac{dx}{dt} = \frac{d}{dt}(P_k x)(t) = P_k \left(\sum_{j=1}^{n} \lambda_j P_j \right) x(t) = \lambda_k (P_k x)(t).$$

From boundedness of $P_k x$ it follows that $\operatorname{Re} \lambda_k = 0$ and $(P_k x)(t) = e^{i\sigma_k t}(P_k x)(0)$, where $\sigma_k = \operatorname{Im} \lambda_k$. Thus,

$$x(t) = \sum_{k=1}^{n} (P_k x)(t) = \sum_{k=1}^{n} e^{i\sigma_k t}(P_k x)(0)$$

is in $AP(X)$. Next we have

PROPOSITION 2. *If $f(t) \in AP(X)$ and $x(t)$ is a solution of the differential equation $dx/dt = Ax + f$ which has relatively compact range, then $x \in AP(X)$.*

Proof. Proceeding as above, we obtain the equality

$$\frac{d}{dt}(P_k x)(t) = \lambda_k (P_k x)(t) + (P_k f)(t).$$

Here $P_k x$ has relatively compact range and $P_k f \in AP(X)$. It follows that $P_k x \in AP(X)$ (see [7]). Thus x is also in $AP(X)$.

Consider now an operator $A \in \mathscr{L}(X)$ which is nilpotent (that is, $A^n = \theta$ for some natural power n). We have

PROPOSITION 3. *Any solution $x(t)$ of the differential equation $x'(t) = Ax(t)$ which is bounded over $\mathbb{R}$ is a constant function.*

Proof. Remark that

$$x(t) = e^{At}x(0) = \left(I + \frac{t}{1!} A + \cdots + \frac{t^{n-1}}{(n-1)!} A^{n-1} \right) x(0),$$

which is a polynomial in t (of degree $n-1$) with coefficients in X. If such a polynomial is bounded over $\mathbb{R}$ it is constant, as easily seen. (If $P(t) = \sum_{k=0}^{m} a_k t^k$, $a_k \in X$, and $\| P(t) \| \leqslant C$ $\forall t \in \mathbb{R}$, it follows that $a_k = \theta$ for $k = 1, 2, \ldots, m$, and $P(t) = a_0 = P(0)$; in fact, if $a_m \neq \theta$, $P(t) = t^m(a_m + (1/t) a_{m-1} + \cdots + (1/t^m) a_0)$ and $\| P(t) \| = |t|^m \| a_m + (1/t) a_{m-1} + \cdots + (1/t^m) a_0 \| \geqslant |t|^m (\| a_m \| - (1/|t|) \| a_{m-1} \| - \cdots - (1/|t|^m) \| a_0 \|)$. For $|t| \geqslant \bar{t}$ it is $(1/|t|) \| a_{m-1} \| + \cdots + (1/|t|^m) \| a_0 \| < \frac{1}{2} \| a_m \|$ and $\| P(t) \| \geqslant (|t|^m/2) \| a_m \|$ so that $\| P(t) \| \to \infty$ as $|t| \to \infty$). Therefore, the bounded solution $x(t)$ which equals $x(0)$ is in $AP(X)$.

We state now a result worked out together with Professor O. Arino (Pau, France):

PROPOSITION 4. *Let $f(t) \in AP(X)$, where X is uniformly convex. Assume the existence of a bounded solution (over $\mathbb{R}$), $x(t)$, of the equation $x'(t) = Ax(t) + f(t)$. Then $x(t)$ is almost-periodic in X.*

Proof. From $x' = Ax + f$ we deduce that $(A^{n-1}x)' = A^{n-1}f$ is a.p., hence $A^{n-1}x$ is a.p. (theorem of Amerio). Next, $(A^{n-2}x)' = A^{n-1}x + A^{n-2}f$ is a.p. and again $A^{n-2}x$ is a.p. Continuing this way we get that x is a.p. If X is a general B-space and $x(t)$ has rel. compact range, one obtains in the same way, by a theorem of Bochner, that $x(t)$ is almost-periodic.

4

In this section we prove a result about almost-periodicity of relatively compact solutions in a situation similar to [2, pp. 125–126] (time-dependent operator). We consider a Hilbert space H and then a family of linear operators $A(t)$, $t \in \mathbb{R}$, having a common linear subspace $\mathscr{D} = \mathscr{D}(A(t))$ $\forall t \in \mathbb{R}$, as domain of definition; also $\mathscr{D}$ is included in H. Thus $A(t)\colon \mathscr{D} \subset H \to H$, $\forall t \in \mathbb{R}$. We have the following

THEOREM 4. *Let* $(A(t)\,h, k)_H = -(h, A(t)\,k)_H$ $\forall h, k \in \mathscr{D}$ *and* $A(t+\omega) = A(t)$, $\forall t \in \mathbb{R}$, *for some real number* ω. *Let* $x(t)$, $\mathbb{R} \to \mathscr{D}$ *be a solution of the differential equation*

$$x'(t) = A(t)\,x(t), \qquad \forall t \in \mathbb{R},$$

such that the set $\{x(t), t \in \mathbb{R}\}$ *has compact closure in* H. *Then* $x(t)$ *is almost-periodic*, $\mathbb{R} \to H$.

We shall use the following criterion of almost-periodicity (see [2, p. 8]). "Let $f(t) \in C_b(\mathbb{R}; X)$, space of continuous bounded functions, $\mathbb{R} \to X$, a Banach space, and its Bochner transform $\tilde{f}(s)$, $\mathbb{R} \to C_b(\mathbb{R}; X)$ be defined by $\tilde{f}(s) = \{f(t+s), t \in \mathbb{R}\}$. Assume that there exists a relatively dense sequence of real numbers $\{s_n\}$, such that the sequence $\{\tilde{f}(s_n)\}$ is relatively compact in $C_b(\mathbb{R}; X)$. Then f is almost-periodic, $\mathbb{R} \to X$."

Proof of the Theorem. If $x(t)$ is a solution we see that $(d/dt)(x(t), x(t)) = (A(t)\,x(t), \; x(t)) + (x(t), A(t)\,x(t)) = 0$ and $\|x(t)\| = \|x(0)\|$ $\forall t \in \mathbb{R}$ (so that $x(t)$ is bounded over $\mathbb{R}$). Remark also that for any integer $n \in \mathbb{Z}$ it is $A(t + n\omega) = A(t)$, $\forall t \in \mathbb{R}$, and

$$\frac{d}{dt}\, x(t+n\omega) = A(t+n\omega)\, x(t+n\omega) = A(t)\, x(t+n\omega).$$

Therefore, if $m \neq n$, we have

$$\frac{d}{dt}\,[x(t+n\omega) - x(t+m\omega)] = A(t)[x(t+n\omega) - x(t+m\omega)]$$

and consequently

$$\|x(t+n\omega)-x(t+m\omega)\|_H = \|x(n\omega)-x(m\omega)\|_H, \qquad \forall t \in \mathbb{R},$$

which, translated in terms of Bochner transform, becomes

$$\|\tilde{x}(n\omega)-\tilde{x}(m\omega)\|_{C_b(\mathbb{R};H)} = \|x(n\omega)-x(m\omega)\|_H.$$

Now, the sequence $\{n\omega\}_{n\in\mathbb{Z}}$ is relatively dense in $\mathbb{R}$, the sequence $\{x(n\omega)\}$ is rel·compact in H by hypothesis, and the above equality shows that the sequence $\{\tilde{x}(n\omega)\}_{n\in\mathbb{Z}}$ is also rel. compact in $C_b(\mathbb{R}; H)$.

5

In this section we consider results concerning separation from θ of non-trivial almost-periodic solutions as in our previous paper [12]. First, a *weakly sequentially complete* Banach space X is given. A function $f(t)$, $\mathbb{R}\to X$ is said to be weakly almost-periodic if $\forall x^* \in X^*$, the function $\langle x^*, f(t)\rangle$, which is scalar-valued, is also (Bohr) almost-periodic. It was proved in [1] that such a function has Bochner's normality property, that is: for any real sequence $\{s_n\}_1^\infty$ one may extract a subsequence $\{s'_n\}_1^\infty$, such that (weak) $\lim_{n\to\infty} f(t+s'_n)=g(t)$ exists, uniformly on $\mathbb{R}$.

Next consider in X a linear closed operator A of dense domain $\mathscr{D}(A)$, which is the infinitesimal generator of a C_0-semi-group T_t. The following is true:

THEOREM 5. *Let $x(t)$, $\mathbb{R}\to\mathscr{D}(A)$ be a solution of $x'(t)=Ax(t)$ which is weakly almost-periodic. Then $x(t)=\theta\ \forall t\in\mathbb{R}$ or $\inf_{t\in\mathbb{R}}\|x(t)\|>0$.*

Proof. If $\inf_{t\in\mathbb{R}}\|x(t)\|=0$, we can find a real sequence $\{\alpha'_n\}\subset\mathbb{R}$ such that $\lim_{n\to\infty} x(\alpha'_n)=\theta$. Therefore, a subsequence $\{\alpha_n\}\subset\{\alpha'_n\}$ exists and a weakly almost-periodic function $y(t)$, $\mathbb{R}\to X$, in such a way that (weak) $\lim_{n\to\infty} x(t+\alpha_n)=y(t)$ exists, uniformly on $\mathbb{R}$. (It implies that (weak) $\lim_{n\to\infty} y(t-\alpha_n)=x(t)$, uniformly on $\mathbb{R}$.)

Because of the representation formula $x(s+\zeta)=T_\zeta x(s)$, $s\in\mathbb{R}$, $\zeta\geq 0$, one deduces formula $x(t+\alpha_n)=T_t x(\alpha_n)$, $\forall t\geq 0$, $n=1, 2,\ldots$. As $x(\alpha_n)\to\theta$ (for $n\to\infty$), we see that $x(t+\alpha_n)\to\theta$, $n\to\infty$ (strongly), $\forall t\in\mathbb{R}^+$. It follows that $\langle x^*, y(t)\rangle=0\ \forall t\geq 0$, hence (from almost-periodicity), for any $t\in\mathbb{R}$, and thus $y(t)=\theta\ \forall t\in\mathbb{R}$ so that $x(t)$ will be null too.

As a second result we also extend here Theorem 3 in [12] to the context of *general* (*non-separable*) Banach spaces. Let X be such a (*general*) B-space and $A(t)$, $\mathbb{R}\to\mathscr{L}(X)$ be an operator-valued function which is strongly almost-periodic, in the sense that $A(t)\,x$, $\mathbb{R}\to X$ is almost-periodic, $\forall x\in X$. Then the following holds:

THEOREM 6. *Let $x(t)$, $\mathbb{R} \to X$ be an almost-periodic solution of the differential equation $x'(t) = A(t)\,x(t)$. Then $x(t) = \theta$ $\forall t \in \mathbb{R}$ or $\inf_{t \in \mathbb{R}} \|x(t)\| > 0$.*

Proof. If we assume that $\inf_{t \in \mathbb{R}} \|x(t)\| = 0$, there exists a real sequence $\{\alpha'_n\}$ such that $x(\alpha'_n) \to \theta$ as $n \to \infty$. On the other hand, using [13] we see that the function $A(t)\,x(t)$ is almost-periodic, $\mathbb{R} \to X$. We may therefore extract a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that $x(t+\alpha_n) \to \tilde{x}(t)$, $A(t+\alpha_n)\,x(t+\alpha_n) \to y(t)$, both limits uniformly on the real line.

Remember now that from the relative compactness of the set $\mathscr{R}x(\cdot) = \{x(t), t \in \mathbb{R}\}$ it follows that the linear closed subspace X_0 of X which is spanned by $\mathscr{R}x(\cdot)$ is a separable one. Consider now the restriction of $A(t)$ to this subspace X_0, which means to consider $A(t)$ as an element of the space $\mathscr{L}(X_0, X)$. With an obvious extension of the Lemma in [12, p. 841], the result expressed there is valid when $A(t) \in \mathscr{L}(E, F)$ where E is a separable and F an arbitrary Banach space. Accordingly, after some more extractions of subsequences, we may also assume that $\lim_{n \to \infty} A(t+\alpha_n)\,x$ exists $\forall x \in X_0$, uniformly on $\mathbb{R}$. Let $B(t)$ be the (linear) operator, $X_0 \to X$, defined by $B(t)\,x = \lim_{n \to \infty} A(t+\alpha_n)\,x$. It is seen that $\|B(t)\,x\|_X \leqslant \sup_{t \in \mathbb{R}} \|A(t)\|_{\mathscr{L}(X_0, X)} \cdot \|x\|$, $\forall x \in X_0$, and that the function $B(t)\,x$ is continuous, $\mathbb{R} \to X$, $\forall x \in X_0$. Moreover, using the relation $\tilde{x}(t) \in X_0$, $\forall t \in \mathbb{R}$, we obtain, as in [12], the relation

$$\lim_{n \to \infty} A(t+\alpha_n)\,x(t+\alpha_n) = B(t)\,\tilde{x}(t), \qquad \forall t \in \mathbb{R},$$

and the uniform bound $\|A(t+\alpha_n)\,x(t+\alpha_n)\| \leqslant C$, $n = 1, 2, \ldots$, because of boundedness of almost-periodic functions. Therefore, applying a property of Bochner's integral, we get, $\forall t \in \mathbb{R}$,

$$\lim_{n \to \infty} \int_0^t A(s+\alpha_n)\,x(s+\alpha_n)\,ds = \int_0^t B(s)\,\tilde{x}(s)\,ds.$$

Remark now that $x'(t+\alpha_n) = A(t+\alpha_n)\,x(t+\alpha_n)$ and

$$x(t+\alpha_n) - x(\alpha_n) = \int_0^t A(s+\alpha_n)\,x(s+\alpha_n)\,ds, \qquad t \in \mathbb{R}.$$

As $n \to \infty$ it results, if $t > 0$,

$$\tilde{x}(t) = \int_0^t B(s)\,\tilde{x}(s)\,ds \quad \text{and} \quad \|\tilde{x}(t)\| \leqslant \int_0^t \|B(s)\,\tilde{x}(s)\|\,ds \leqslant L \int_0^t \|\tilde{x}(s)\|\,ds$$

where $L = \sup_{s \in R} \|B(s)\|_{\mathscr{L}(X_0, X)}$, and $t > 0$. Defining the function

$$\sigma(t) = \int_0^t \|\tilde{x}(s)\|\,ds, \qquad t \geqslant 0,$$

we see that $\sigma'(t) = \|\tilde{x}(t)\| \leqslant L\sigma(t)$ and $(d/dt)(e^{-Lt}\sigma(t)) \leqslant 0$, $e^{-Lt}\sigma(t) \leqslant 0$, $t > 0$, therefore $\sigma(t) = 0$ for $t > 0$ and $\|\tilde{x}(t)\| = 0 \; \forall t \geqslant 0$. Because of the almost-periodicity of $\tilde{x}$, it is null for any real t and the same is obviously true for $x(t)$, which equals $\lim_{n \to \infty} \tilde{x}(t - \alpha_n)$.

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